Projective varieties II : Ringed spaces

Big picture
- Have defined projective varieties $X \subseteq \mathbb{P}^n$ as sets & top. spaces
- Now: define sheaf $\mathcal{O}_X$ of regular functions
  $\rightsquigarrow (X, \mathcal{O}_X)$ ringed space $\rightsquigarrow$ prevariety $\rightsquigarrow$ variety

X affine, $U \subseteq X$ open $\rightsquigarrow \varphi: U \to K$ regular if locally $\varphi(x) = \frac{g(x)}{f(x)}$
$f, g \in A(X)$

X projective $\rightsquigarrow$ have homogeneous coord. ring $S(X)$,
but $g(x)/f(x)$ does not always make sense

Exa $g(x_0, x_1) = x_0^2$, $f(x_0, x_1) = x_1 \longrightarrow \varphi(x_0, x_1) = g(x_0, x_1)/f(x_0, x_1)$ has
$\varphi(1,1) = 1 \neq \varphi(2,2) = 2$.
not invariant under scaling

Solution: require f, g to be of same degree d!

Def (Regular functions on projective varieties)
X projective variety, $U \subseteq X$ open
$\rightsquigarrow \varphi: U \longrightarrow K$ is a <u>regular function</u> if:

For all $a \in U$ there are $d \in \mathbb{N}$, $f, g \in S(X)_d$ w/ $f(a) \neq 0$ st.

$$\varphi(x) = \frac{g(x)}{f(x)} \quad \text{for all } x \in U_a$$

some open subset $a \in U_a \subseteq X$.

Write $\mathcal{O}_X(U)$ for set of regular functions on U.

Rmk • $\mathcal{O}_X(U)$ is K-algebra : $f_j, g_j \in S(X)_{d_j}$ $(j = 1,2)$

$$\rightsquigarrow \frac{g_1(x)}{f_1(x)} + \frac{g_2(x)}{f_2(x)} = \frac{(g_1 \cdot f_2 + g_2 \cdot f_1)(x)}{(f_1 \cdot f_2)(x)}$$

homog. degree $d_1 + d_2$

(similar for $\cdot$)

• condition on reg. funct. is local $\Rightarrow \mathcal{O}_X$ is sheaf on X

Result $X \subseteq \mathbb{P}^n$ proj. variety $\rightsquigarrow$ $(X, \mathcal{O}_X)$ ringed space
Next: Check that this is a prevariety.

Idea: $\mathbb{P}^n = U_0 \cup \dots \cup U_n$ affine cover
$\rightsquigarrow X = (X \cap U_0) \cup \dots \cup (X \cap U_n)$ affine cover.

Pro (Projective varieties are prevarieties)
$X \subseteq \mathbb{P}^n$ projective variety, then

$$U_i = \{(x_0 : \dots : x_n) \in X : x_i \neq 0\} \subseteq X$$

is an affine variety for $i = 0, \dots, n$.
In particular: X is a prevariety.

Proof By symmetry: suffices $i = 0$.
$X = V_p(J)$ for $J \trianglelefteq K[x_0, \dots, x_n]$ homogeneous
Let $Y := V_a(J^i) \subseteq \mathbb{A}^n$.
Claim The map $\rightsquigarrow$ Claim finishes proof. ($X = U_0 \cup \dots \cup U_n$ cover by affine varieties)

$$\underset{\mathcal{O}_Y \text{ from } Y \text{ affine var.}}{F : Y} \longrightarrow \underset{\mathcal{O}_{U_0} = \mathcal{O}_X|_{U_0}}{U_0}, \quad (x_1, \dots, x_n) \longmapsto (1 : x_1 : \dots : x_n)$$

is an isomorphism with inverse

$$F^{-1} : U_0 \longrightarrow Y, \quad (x_0 : \dots : x_n) \longmapsto \left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right).$$

Proof of claim. • F, F^{-1} well-defined maps of sets, inverse pair
• Continuous: $F^{-1}(\underbrace{V_p(J') \cap U_0}_{\text{closed in } U_0}) = \underbrace{V_{a,Y}((J')^i)}_{\text{closed in } Y}$ [Rmk. 6.27]

$$F(V_a(\tilde{J})) = V_p(\tilde{J}^h) \cap U_0$$

- Pull back regular fcts. to regular fcts.

φ on (open subset of) U_0 regular $\Leftrightarrow$ φ locally $\dfrac{g(x_0,\dots,x_n)}{f(x_0,\dots,x_n)}$ $f,g\in S(X)_d$ homogen.

$\rightsquigarrow F^* \dfrac{g(x_0,\dots,x_n)}{f(x_0,\dots,x_n)} = \dfrac{g^i(x_1,\dots,x_n)}{f^i(x_1,\dots,x_n)}$ regular on Y

<u>Conversely</u>

φ on (open subset of) Y regular $\Leftrightarrow$ φ locally $\dfrac{g(x_1,\dots,x_n)}{f(x_1,\dots,x_n)}$ $f,g\in A(X)$

$\rightsquigarrow (F^{-1})^* \dfrac{g(x_1,\dots,x_n)}{f(x_1,\dots,x_n)} = \dfrac{g\left(\frac{x_1}{x_0},\dots,\frac{x_n}{x_0}\right)\cdot x_0^e}{f\left(\frac{x_1}{x_0},\dots,\frac{x_n}{x_0}\right)\cdot x_0^e}$ for $e=\max(\deg f,\deg g)$

Check: numerator & denominator homogeneous of degree e.

$\Rightarrow F, F^{-1}$ morphisms, inverse to each other. $\square$

<u>Exercise</u> (Compatibility checks)

(a) Prevariety $\mathbb{P}^1$ above = Prevariety $\mathbb{P}^1$ from gluing [Exa 5.5(a)]

(b) $X\subseteq\mathbb{P}^n$ proj. variety = closed subvariety $X\subseteq\mathbb{P}^n$.

Morphisms of projective varieties

Advantage of our approach over gluing:

- no compatibility checks on overlaps
- many morphisms have global description

Lem (Morphisms of projective varieties)

$X \subseteq \mathbb{P}^n$ projective variety, $f_0, \dots, f_m \in S(X)$ homogen. of same degree

$\rightsquigarrow$ For $U = X \setminus V_p(f_0, \dots, f_m) \subseteq X$ we have a morphism

$$f : U \longrightarrow \mathbb{P}^m, \quad x \mapsto (f_0(x) : \dots : f_m(x)).$$

Proof

- well-defined map of sets:
  - $\to$ $(f_0(x), \dots, f_m(x)) \neq (0, \dots, 0)$ by def of U
  - $\to$ independent of scaling x since f_i homog. of same degree

- morphism: gluing property of morphisms [Lem 4.6]

  $\rightsquigarrow$ suff. to check on open cover

  Let $V_i = \{(y_0 : \dots : y_m) \in \mathbb{P}^m : y_i \neq 0\}$ be open cover of $\mathbb{P}^m$

  $\rightsquigarrow U_i = f^{-1}(V_i) = \underbrace{\{x \in X : f_i(x) \neq 0\}}_{= U \setminus V_p(f_i) \text{ open in } U.}$ open cover of U

$$\begin{tikzcd} U_i \arrow[r, "f"] \arrow[dr, "f^{(i)}"] & V_i \arrow[d, "F^{-1}", "\cong"'] \\ & \mathbb{A}^m \end{tikzcd}$$

$$x \mapsto \left(\frac{f_0}{f_i}, \dots, \frac{f_m}{f_i}\right)$$

coordin. = quotients of polynomials = reg. funct. on U_i

$\Rightarrow f^{(i)}$ morphism [Pro 4.7] $\square$

note: proved this when domain is affine variety, holds more gen. for prevariety.

Exq

(a) $A \in GL(n+1, K)$ ← invertible $(n+1)\times(n+1)$ matrices w/ entries in K.

$\rightsquigarrow$ $\varphi: \mathbb{P}^n \longrightarrow \mathbb{P}^n$, $x \longmapsto Ax$ is (iso) morphism with inverse
$\varphi^{-1}: \mathbb{P}^n \longrightarrow \mathbb{P}^n$, $y \longmapsto A^{-1}y$.

$Ax = (\sum a_{ij}x_j)_{i=1}^n$ homogr. degree 1.

Call these maps <u>projective automorphisms of</u> $\mathbb{P}^n$

<u>Fact</u>: Every automorphism of $\mathbb{P}^n$ is of this form.
Let $PGL(n+1, K) = GL(n+1, K)/\sim$ with $A \sim \lambda A$ for $\lambda \in K^*$

$PGL(n+1,K)$ ← projective general linear group

$\rightsquigarrow$ group with multiplication $[A]\cdot[B] = [A\cdot B]$

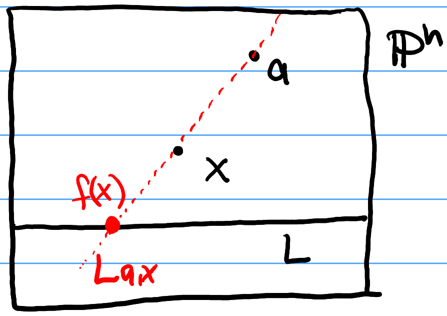
$\rightsquigarrow$ $\mathrm{Aut}(\mathbb{P}^n) = \{\varphi: \mathbb{P}^n \to \mathbb{P}^n : \varphi \text{ isomorph.}\} \cong PGL(n+1, K)$.

Projecting from a point

⇝ continue list of examples of morphisms of proj. varieties

Exa

(b) Projection from a point



$a=(1:0:\dots:0)\in\mathbb{P}^n$, $L=V_P(x_0)\cong\mathbb{P}^{n-1}$

⇝ $f:\mathbb{P}^n\setminus\{a\}\longrightarrow\mathbb{P}^{n-1}, (x_0:\dots:x_n)\mapsto(x_1:\dots:x_n)$

↑ forget x_0

is a morphism [Lem]

Interpretation as projection (picture)

$x=(x_0:\dots:x_n)\in\mathbb{P}^n\setminus\{a\}$

⇝ Unique line L_{ax} through a,x:

$$\begin{array}{ccc} \mathrm{Span}_K(a,x)\setminus\{0\} & \subseteq & K^{n+1}\setminus\{0\} \\ \downarrow & & \downarrow \\ L_{ax} & \subseteq & \mathbb{P}^n \end{array}$$

$$L_{ax}=\{(s:tx_1:\dots:tx_n) : (s:t)\in\mathbb{P}^1\}$$

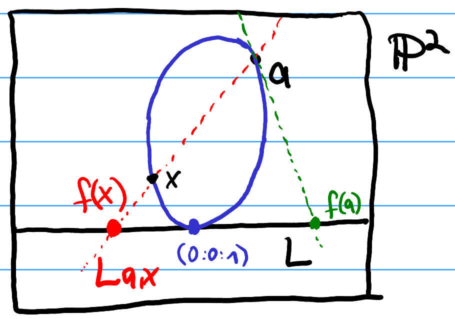
$$L\cap L_{ax}=\{(0:x_1:\dots:x_n)\}$$

↑ $f(x)$ after $L\cong\mathbb{P}^{n-1}$

⇝ f = projection from a to linear subspace L

⌈Works for general a, L w/ $a\notin L$ ⇝ Can always move a, L to ones above by projective automorph.⌋

(c) Projecting from a conic



Projection morphism

$f:\mathbb{P}^n\setminus\{a\}\longrightarrow\mathbb{P}^{n-1}$

does not extend to $\{a\}$.

$\lim_{x\to a} f(x)$ depends on how $x\to a$.

However: can extend f on some $X\subseteq\mathbb{P}^n$ subvar. containing a

$$X=V_P(x_0x_2-x_1^2)\ni a=(1:0:0)$$

$$\varphi: X\longrightarrow\mathbb{P}^1,\quad (x_0:x_1:x_2)\longmapsto\begin{cases}(x_1:x_2) & \text{if } (x_0:x_1:x_2)\neq a\\ (x_0:x_1) & \text{if } (x_0:x_1:x_2)\neq(0:0:1)\end{cases}$$

Properties of f:

- Well-defined: $X_0X_2 = X_1^2 \rightsquigarrow \frac{X_1}{X_0} = \frac{X_2}{X_1}$ (if $x_0, x_1 \neq 0$)

  $\Rightarrow (x_1 : x_2) = (x_0 : x_1)$ $\rightsquigarrow$ definit. agree on overlap $X \setminus \{a, (0:0:1)\}$.

  Scale by x_1/x_0

- Morphism ([Lem])
- Extends the map from (b) to all of X
- Check: $f(a)$ is intersection point of L and tangent to X at a
- Bijection:

  $y \in L \rightsquigarrow$ line $L_{a,y} \cong \mathbb{P}^1$ from a to y

  $\rightsquigarrow X_0 \cdot X_2 - X_1^2 \big|_{L_{a,y}}$ = polynomial g of degree 2

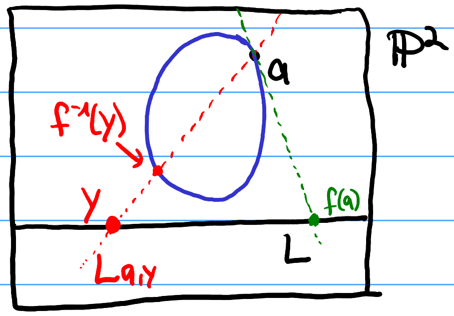
Exercise $f \in K[x_0, x_1] = S(\mathbb{P}^1)$ non-zero, homogeneous of degree d

$\Rightarrow f$ has d zeros on $\mathbb{P}^1$, counted with multiplicity.

computed on affine patches U_0, U_1

$\Rightarrow g$ has two zeros, one at $x = a$, other at $f^{-1}(y)$

$\Rightarrow f$ is bijective



$f^{-1}: \mathbb{P}^1 \to X, (y_0 : y_1) \mapsto (y_0^2 : y_0 y_1 : y_1^2)$

(d) $X \subseteq \mathbb{P}^2$ irreducible quadric curve $\leftarrow$ projective conic

$\xrightarrow[\text{[Exerc. 4.12]}]{\text{char} K \neq 2}$ $X \cap \mathbb{A}^2 \cong V_a(x_2 - x_1^2)$ or $V_a(x_1x_2 - 1)$ $(*)$

via linear transformat. + translation $\mathbb{A}^2 \xrightarrow{T} \mathbb{A}^2$

Note T extends to projective automorph. $\mathbb{P}^2 \to \mathbb{P}^2$ as in (a)

check in coordinates!

$(*) \Rightarrow X = \overline{X \cap \mathbb{A}^2} \cong V_p(x_0x_2 - x_1^2)$ or $V_p(x_1x_2 - x_0^2)$

projective closure of $V_a(f) = V_p(f^h)$.

$\Rightarrow X$ isom. to $V(x_0x_2 - x_1^2)$ from (c) $\Rightarrow X \cong \mathbb{P}^1$.

Slogan: Every irred. proj. conic is isomorph. to $\mathbb{P}^1$.

<u>Note</u> Example (c)

$$\varphi : X \longrightarrow \mathbb{P}^1, \quad (x_0 : x_1 : x_2) \longmapsto \begin{cases} (x_1 : x_2) & \text{if } (x_0 : x_1 : x_2) \neq a \\ (x_0 : x_1) & \text{if } (x_0 : x_1 : x_2) \neq (0:0:1) \end{cases}$$

$\Rightarrow$ Not every morphism $\underset{\text{proj. var.}}{X} \to \mathbb{P}^n$ is given by $(n+1)$ polynom

<u>Exercise</u> Every morphism $\varphi : \mathbb{P}^n \to \mathbb{P}^m$ is given by

$$\varphi(x) = (f_0(x), \ldots, f_m(x)), \quad f_i \text{ homog. of same degree.}$$

$\rightsquigarrow$ this can be used to show $\mathrm{Aut}(\mathbb{P}^n) = PGL(n+1, K)$!

The Segre embedding

Big picture

Suffices: $X = \mathbb{P}^n$

→ Proved: projective varieties are prevarieties
→ For separatedness: understand diagonal $\Delta_{\mathbb{P}^n} \subseteq \mathbb{P}^n \times \mathbb{P}^n$
→ Below: embed $\mathbb{P}^n \times \mathbb{P}^n$ into bigger proj. space
⇝ coordinates to write down equat. of $\Delta_{\mathbb{P}^n}$.

Construction (Segre embedding)

Consider $\mathbb{P}^n$ (coord. $x_0, \dots, x_n$) and $\mathbb{P}^m$ (coord. $y_0, \dots, y_m$)
Set $N := (n+1)(m+1) - 1$ ⇝ $\mathbb{P}^N$ (coord. z_{ij}, $i = 0, \dots, n$, $j = 0, \dots, m$)

$$\rightsquigarrow f : \mathbb{P}^n \times \mathbb{P}^m \longrightarrow \mathbb{P}^N, \quad ((x_i)_{i=0}^n, (y_j)_{j=0}^m) \mapsto (z_{ij} = x_i \cdot y_j)_{\substack{i=0,\dots,n \\ j=0,\dots,m}}$$

well-defined set-theoretic map.

Pro The map f above satisfies:
(a) The image $X = f(\mathbb{P}^n \times \mathbb{P}^m) \subseteq \mathbb{P}^N$ is a projective variety given by

$$X = V_p(z_{ij} z_{kl} - z_{il} z_{kj} : 0 \le i, k \le n, 0 \le j, l \le m).$$

(b) The map $f : \mathbb{P}^n \times \mathbb{P}^m \to X$ is an isomorphism.
⇝ f is a closed embedding.

Proof (a) Points in image of f satisfy conditions:
$z_{ij} z_{kl} - z_{il} z_{kj} = x_i y_j \cdot x_k y_l - x_i y_l x_k y_j = 0$.

Conversely, assume $(z_{ij})_{ij}$ satisfies the conditions
⇝ one $z_{ij} \neq 0$ ⇝ Wlog. $z_{00} \neq 0$

⇒ Look at affine chart $U_{00} = \{(z_{ij})_{ij} : z_{00} \neq 0\} \subseteq \mathbb{P}^N$
& study f in affine coordinates ← set $z_{00} = 1$

$\forall i,j$ have equations $1 \cdot z_{ij} = \underbrace{z_{i,0}}_{=:x_i} \cdot \underbrace{z_{0,j}}_{=:y_j}$

$\Rightarrow (z_{ij})_{ij} = f((x_i)_i, (y_j)_j)$

<u>Note</u> $f^{-1}(U_{00}) = \underset{\{x_0 \neq 0\}}{U_0^{\mathbb{P}^n}} \times \underset{\{y_0 \neq 0\}}{U_0^{\mathbb{P}^m}}$

In affine coordinates:

$f(x_1, \ldots, x_n, y_1, \ldots, y_m) = (\underbrace{x_1, \ldots, x_n}_{z_{i,0}}, \underbrace{y_1, \ldots, y_m}_{z_{0,j}}, \underbrace{(x_i y_j)}_{z_{ij}}{}_{\substack{i=1,\ldots,n \\ j=1,\ldots,m}})$ $i=1,\ldots,n, j=1,\ldots,m$

Thus:
- f is injective with inverse f^{-1} = project. on first $n+m$ coordinates
- f, f^{-1} morphisms
  given by polynomials in affine coordinates.

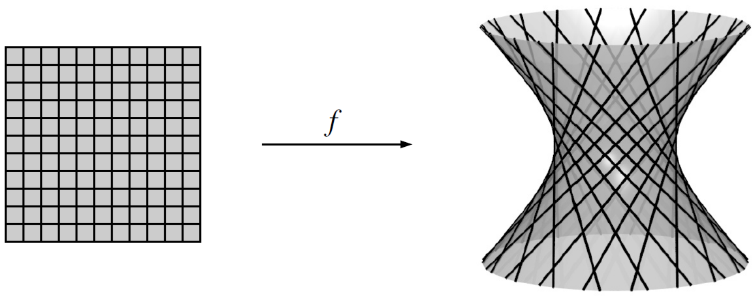
$\square$

<u>Ex9</u> (Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$)
$\mathbb{P}^1 \times \mathbb{P}^1$ is isomorphic to

$$X = \{(z_{00} : z_{01} : z_{10} : z_{11}) : z_{00} z_{11} - z_{10} \cdot z_{01} = 0\} \subseteq \mathbb{P}^3$$

via the Segre embedding:

$$f : \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow X, \quad ((x_0 : x_1), (y_0 : y_1)) \mapsto (x_0 y_0 : x_0 y_1 : x_1 y_0 : x_1 y_1)$$



$\mathbb{P}^1 \times \mathbb{P}^1$ $\quad$ $X \subset \mathbb{P}^3$

Cor Every projective variety is a variety.

Pf Know: every proj. variety is closed subprevariety of some $\mathbb{P}^n$

$\rightsquigarrow$ suffices: $\mathbb{P}^n$ separated

$$\Delta_{\mathbb{P}^n} = \{((x_0 : \dots : x_n), (y_0 : \dots : y_n)) : x_i y_j - x_j y_i = 0 \ \forall i,j\}$$

all 2×2 minors of $\begin{pmatrix} x_0 & \dots & x_n \\ y_0 & \dots & y_n \end{pmatrix}$ vanish

$\overset{\text{Lin Alg}}{\Longleftrightarrow}$ rows of matrix are lin. dependent

$\Longleftrightarrow (x_0 : \dots : x_n) = (y_0 : \dots : y_n)$.

$$\Rightarrow \mathbb{P}^n \times \mathbb{P}^n \xrightarrow{f} \mathbb{P}^N$$

$$\cup\!|$$

$$\Delta_{\mathbb{P}^n} = f^{-1}(V_P(\underbrace{z_{ij} - z_{ji}}_{\substack{\text{linear polyn.} \\ \text{in Segre coordinates}}} : i,j = 0, \dots, n)) \text{ closed. } \square$$

Rmk

$X \subseteq \mathbb{P}^n, Y \subseteq \mathbb{P}^m$ projective

$$\rightsquigarrow X \times Y \xhookrightarrow[\text{embedd.}]{\text{closed}} \mathbb{P}^n \times \mathbb{P}^m \xhookrightarrow[\text{embedding}]{\text{Segre}} \mathbb{P}^N$$

variety

$\Rightarrow X \times Y$ projective variety

Closed maps

Big picture

- One motivation for studying projective space: $\mathbb{P}^n_{\mathbb{C}}$ compact $\rightsquigarrow$ $X \subseteq \mathbb{P}^n_{\mathbb{C}}$ proj. variety also compact.
  closed also in $\mathbb{C}$-topology
- Zariski-topology: every prevariety is compact [Ex. 2.36(b)]
- Try to generalize other nice properties of compact sets (image of cpt. = compact, closed subset of cpt. = cpt.)

Def (Closed maps)
A map $f: X \to Y$ between top. spaces is <u>closed</u> if for all $A \subseteq X$ closed also $f(A) \subseteq Y$ is closed.

Want to show: X proj. var., Y var. $\Rightarrow$ any morphism $f: X \to Y$ is closed.
$\rightsquigarrow$ start w/ special case.

Pro The projection map $\pi: \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^m$ is closed.

Proof $Z \subseteq \mathbb{P}^n \times \mathbb{P}^m \xhookrightarrow{\text{Segre}} \mathbb{P}^N$
closed   Segre coordinates z_{ij}

$\rightsquigarrow Z = V_{\mathbb{P}^n \times \mathbb{P}^m}(f_1, \dots, f_r)$, f_k homogeneous in $(z_{ij})_{ij}$

Without loss of generality: all of same degree d [Rmk 6.23]

For coord. $x_0, \dots, x_n$, $y_0, \dots, y_m$ on $\mathbb{P}^n$, $\mathbb{P}^m$:
$f_k(-,-)$ bihomogeneous of degree d in x_i, y_j.

Summary $Z \subseteq \mathbb{P}^n \times \mathbb{P}^m \xrightarrow{\pi} \mathbb{P}^m$ $\rightsquigarrow$ Show: $\pi(Z)$ closed.

$\overset{\parallel}{V_{\mathbb{P}^n \times \mathbb{P}^m}(f_1, \ldots, f_r)}$

bihomog. of deg. d in (x, y)

Fix $a \in \mathbb{P}^m$ $\rightsquigarrow$ Is $a \in \pi(Z)$?

Let $g_i = f_i(-, a) \in K[x_0, \ldots, x_n]$ for $i = 1, \ldots, r$

unique up to scaling

$$\begin{aligned}
a \notin \pi(Z) &\iff \text{there is no } x \in \mathbb{P}^n : (x, a) \in Z \\
&\iff V_P(g_1, \ldots, g_r) = \emptyset \qquad \text{[Pro 6.20]} \\
&\iff \sqrt{\langle g_1, \ldots, g_r \rangle} = \langle 1 \rangle \text{ or } \sqrt{\langle g_1, \ldots, g_r \rangle} = \langle x_0, \ldots, x_n \rangle \\
&\iff \exists\, k_i \in \mathbb{N} : x_i^{k_i} \in \langle g_1, \ldots, g_r \rangle \\
&\iff K[x_0, \ldots, x_n]_k \subseteq \langle g_1, \ldots, g_r \rangle \text{ for some } k \in \mathbb{N} \qquad (*)
\end{aligned}$$

"$\Rightarrow$" take $k = k_0 + \ldots + k_n$

homogeneous degree k part.

$\rightsquigarrow (*) \iff d \leq k$ and $K[x_0, \ldots, x_n]_k = \langle g_1, \ldots, g_r \rangle_k$

form of general element

$\iff F_k : (K[x_0, \ldots, x_n]_{k-d})^r \to K[x_0, \ldots, x_n]_k, (h_1, \ldots, h_r) \mapsto h_1 g_1 + \ldots + h_r g_r$ surjective for some $k \geq d$

K-linear map, target has $\dim_K = \binom{n+k}{k}$

F_k surjective $\iff$ $\operatorname{rank} F_k = \binom{n+k}{k}$

F_k described by $\binom{n+k}{k} \times \left(r \cdot \binom{n+k-d}{k-d}\right)$ matrix

$\rightsquigarrow$ entries = coeff. of g_i = homog. poly. in coord. of $a \in \mathbb{P}^m$

$\operatorname{rank} F_k = \binom{n+k}{k} \iff$ one of the $\underbrace{\text{minors of size } \binom{n+k}{k}}_{\text{homog. poly in } a}$ of (the matrix of) F_k is nonzero. $\rightsquigarrow$ Zariski open in $\mathbb{P}^m$

$\rightsquigarrow \{a \in \mathbb{P}^m : a \notin \pi(Z)\} \subseteq \mathbb{P}^m$ Zar. open $\iff \pi(Z)$ closed. $\square$

Rmk Equations of $\pi(Z)$ in $y_0, \dots, y_m$

$=$ Equations in $x_0, \dots, x_n, y_0, \dots, y_m$ vanishing on Z that only depend on $y_0, \dots, y_m$ $\leadsto$ eliminate variables x_i

$\leadsto$ Pro. = main theorem of elimination theory.

Cor $\pi : \mathbb{P}^n \times Y \longrightarrow Y$ is closed for any variety Y.

Proof

Step 1 $Y \subseteq \mathbb{A}^m$ affine.

$$\begin{array}{ccccccc} Z \overset{\text{closed}}{\subseteq} \mathbb{P}^n \times Y & \subseteq & \mathbb{P}^n \times \mathbb{A}^m & \subseteq & \mathbb{P}^n \times \mathbb{P}^m \\ \downarrow \pi & & & & \downarrow \pi' \\ Y & \subseteq & \mathbb{A}^m & \subseteq & \mathbb{P}^m \end{array}$$

$\overline{Z}$ = closure of Z in $\mathbb{P}^n \times \mathbb{P}^m \overset{\text{Pro}}{\Longrightarrow} \pi'(\overline{Z}) \subseteq \mathbb{P}^m$ closed

$\Rightarrow \pi(Z) \underset{\substack{Z \text{ closed} \\ \text{in } \mathbb{P}^n \times Y}}{=} \pi'(\overline{Z} \cap \underbrace{(\mathbb{P}^n \times Y)}_{=(\pi')^{-1}(Y)}) = \pi'(\overline{Z}) \cap Y$ closed in Y.

Step 2 Y any variety $\leadsto$ cover by affine opens Y_i

$$\begin{array}{ccc} Z \overset{\text{closed}}{\subseteq} \mathbb{P}^n \times Y & \supseteq & \mathbb{P}^n \times Y_i \\ \downarrow \pi & & \downarrow \\ Y & \supseteq & Y_i \end{array}$$

Step 1: $\pi(Z) \cap Y_i$ closed in Y_i

being closed can be checked on open cover

$\pi(Z)$ closed in Y. $\square$

Complete varieties

Have seen: $\pi : \mathbb{P}^n \times Y \to Y$ closed for any variety Y.

$\rightsquigarrow$ this gives nice analogue of classic. compactness of $\mathbb{P}^n$

Def (Complete varieties)
A variety X is called complete if $\pi : X \times Y \to Y$ is closed for all varieties Y.

Exa

(a) $\mathbb{P}^n$ is complete (see above)

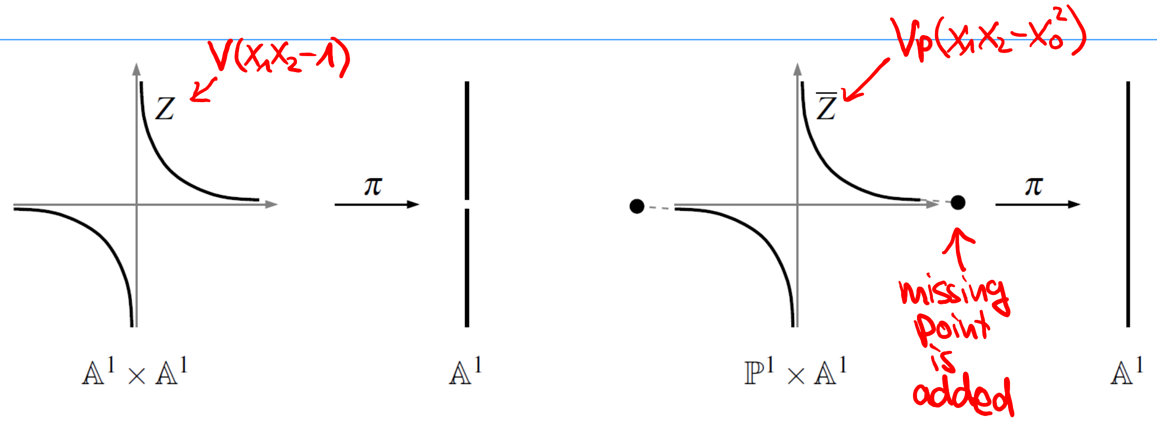
(b) A closed subvar. $X' \subseteq X$ of a complete variety X is complete:

$$\begin{tikzcd} Z \subseteq X' \times Y \arrow[r, hook, "\text{closed}", "\text{emb.}"'] \arrow[d, "\pi'"'] & X \times Y \arrow[d, "\pi"] \\ Y \arrow[r, equal, "\text{id}"] & Y \end{tikzcd}$$

$\rightsquigarrow$ Z closed in $X \times Y$ $\overset{X \text{ complete}}{\Longrightarrow}$ $\pi'(Z) = \pi(Z)$ closed in Y. ※

$\rightsquigarrow$ Every projective variety is complete

(c) $\mathbb{A}^1$ is not complete:



$Z \subseteq \mathbb{A}^1 \times \underbrace{\mathbb{A}^1}_{=Y}$ closed

$\downarrow \pi$

$\mathbb{A}^1 \setminus \{0\} \subseteq \mathbb{A}^1$ not closed.

Rmk $\exists$ complete varieties which are not projective
[Nagata] $\rightsquigarrow$ complicated construction

Cor $f: X \to Y$ morphism of varieties, X complete
$\Longrightarrow$ f closed and $f(X) \subseteq Y$ is complete closed subvar.

Pf

$\pi: X \times Y \to Y$ closed and $\Gamma_f = \{(x, f(x)) : x \in X\} \subseteq X \times Y$ closed

Y variety [Pro 5.20(a)]

$\Rightarrow f(X) = \pi(\Gamma_f) \subseteq Y$ closed subvariety.

For $f(X)$ complete: Y' variety,

$$\begin{tikzcd}
\psi^{-1}(Z) \subseteq X \times Y' \arrow[d, "\psi = (f, \mathrm{id}_{Y'})"] \arrow[dd, bend left=60, "\pi"] \\
Z \subseteq f(X) \times Y' \text{ closed} \arrow[d, "\pi'"] \\
\pi'(Z) \subseteq Y'
\end{tikzcd}$$

$\pi'(Z) \overset{\psi \text{ surj.}}{=} \pi'(\psi(\psi^{-1}(Z)))$

$= \underset{\substack{\text{closed} \\ \text{map } (X \text{ compl.})}}{\pi}(\underbrace{\psi^{-1}(Z)}_{\substack{\text{closed set} \\ (\psi \text{ morphism})}})$ closed ✓

For f closed

$V \subseteq X$ closed $\overset{\text{Exa (b)}}{\Longrightarrow}$ V complete $\Rightarrow f(V) = \mathrm{im}(V \to X \xrightarrow{f} Y)$ closed by first part. $\square$

Exercise (Importance of separated target)

(a) Generalize the construction of the affine line with two origins by gluing two copies of $\mathbb{P}^1$ along $\mathbb{P}^1 \setminus \{0\}$.
$\rightsquigarrow$ projective line X with two origins

(b) Show that the image of $\mathbb{P}^1 \to X$ is not closed in X.
first copy of $\mathbb{P}^1$

Application Complex analysis: holomorphic fct. $\mathbb{P}^1_{\mathbb{C}} \xrightarrow{f} \mathbb{C}$ bounded ($\mathbb{P}^1_{\mathbb{C}}$ compact)
$\Longrightarrow$ By Liouville's thm: f constant, since $f|_{\mathbb{C}}$ bounded entire fct.
$\rightsquigarrow$ below we see algebraic analogue

Cor X connected complete variety $\Rightarrow \mathcal{O}_X(X) = K$

every global regular function is constant.

Pf $\varphi \in \mathcal{O}_X(X) \rightsquigarrow \varphi : X \to \mathbb{A}^1$

Using $\mathbb{A}^1 \subseteq \mathbb{P}^1 \rightsquigarrow \varphi : X \to \mathbb{P}^1 = \mathbb{A}^1 \cup \{\infty\}$ w/ $\infty \notin \varphi(X)$.

Cor. above $\Rightarrow \varphi(X) \subseteq \mathbb{P}^1$ closed $\xRightarrow[\text{of aff. var. in } \mathbb{A}^1]{\text{Classification}} \varphi(X)$ finite or $\underbrace{\varphi(X) = \mathbb{A}^1}_{\text{can exclude}}$

X connected $\xRightarrow{\text{Topology}} \varphi(X)$ connected $\Longrightarrow \varphi(X) = \{a\} \Rightarrow \varphi$ constant.

$\square$

The Veronese embedding

Have seen: $f \in K[x_0,\dots,x_n] = S(\mathbb{P}^n)$ homogeneous of degree 1

$\Rightarrow X = V_P(f)$ linear hypersurface (= hyperplane)

Can check: $\exists$ isomorphism $\mathbb{P}^n \to \mathbb{P}^n$, $X \to V_P(x_0)$ = plane at ∞

$\Rightarrow \mathbb{P}^n \setminus V_P(f) \cong \mathbb{P}^n \setminus V_P(x_0) = \mathbb{A}^n$ is affine.

$\rightsquigarrow$ This holds more generally!

Construction (Veronese embedding)

Let $n, d \in \mathbb{N}_{>0}$ and $f_0, \dots, f_N \in K[x_0,\dots,x_n]$ the set of degree d monomials

$N = \binom{n+d}{n} - 1$

Exa $n=3, d=2 \rightsquigarrow f_0 = x_0^2, f_1 = x_0x_1, f_2 = x_0x_2, f_3 = x_1^2, f_4 = x_1x_2, f_5 = x_2^2$

Consider the map

$$F = F_{n,d} : \mathbb{P}^n \longrightarrow \mathbb{P}^N, \quad x \longmapsto (f_0(x) : \dots : f_N(x))$$

Note: $V_P(f_0,\dots,f_N) \subseteq V_P(x_0^d, \dots, x_n^d) = \emptyset$ $\rightsquigarrow$ Veronese embedding

$\xRightarrow{\text{[Lem 7.4]}}$ F is a morphism defined on all of $\mathbb{P}^n$.

$\left.\begin{array}{l}\mathbb{P}^n \text{ complete variety} \\ \mathbb{P}^N \text{ variety}\end{array}\right\} \Rightarrow X = F(\mathbb{P}^n) \subseteq \mathbb{P}^N$ projective variety.

Claim $F : \mathbb{P}^n \to X$ is an isomorphism. $\rightsquigarrow$ F closed embedding

Pf Write down inverse morphism on open cover:

assume: $\boxed{f_i = x_i^d}$

$$U_i = \{x \in \mathbb{P}^n : x_i \neq 0\} \xrightarrow{F} V_i = \{y \in X : y_i \neq 0\} \rightsquigarrow V_i \subseteq X \text{ open}$$

$y_i \circ F = f_i = x_i^d$

$\cong$

$\mathbb{A}^n$ coordinates $x_0,\dots,x_{i-1},x_{i+1},\dots,x_n$ setting $x_i = 1$

Inverse map: $x_j = \dfrac{x_j x_i^{d-1}}{x_i^d}$ $\leftarrow$ coordinate functions in $S(X)$ $\leftarrow$ denominator vanishes nowhere on V_i

$\rightsquigarrow F$ injective and F^{-1} is a morphism (check on $X = V_0 \cup \dots \cup V_n$).

Rmks

→ F realizes $\mathbb{P}^n$ as subvariety of bigger space $\mathbb{P}^N$

→ Advantage: via F we can relate

$$\left\{\begin{array}{l}\text{degree } d \text{ polynomials}\\ \text{in coord. on } \mathbb{P}^n\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{linear polynomials}\\ \text{in coord. on } \mathbb{P}^N\end{array}\right\}$$

↑ Veronese coordinates

→ $Y \subseteq \mathbb{P}^n$ projective $\Rightarrow F|_Y : Y \to \mathbb{P}^N$ also closed embedding.

Exa

(a) $d=1 \rightsquigarrow F_{n,1} : \mathbb{P}^n \to \mathbb{P}^n$ is identity

(b) $n=1 \rightsquigarrow F_{1,d} : \mathbb{P}^1 \to \mathbb{P}^d$ rational normal curve of degree d

$$(x_0:x_1) \longmapsto (x_0^d : x_0^{d-1}x_1 : \dots : x_0 x_1^{d-1} : x_1^d)$$

Exercise

(a) Find explicit equations describing $F_{n,d}(\mathbb{P}^n) \subseteq \mathbb{P}^N$.

Hint: Possible to find quadratic eqns!

(b) Show that any proj. variety is isomorphic to the zero locus of quadratic polynomials in some proj. space.

Cor $X \subseteq \mathbb{P}^n$ projective variety, $f \in S(X)$ homogen., non-constant. Then $X \setminus V(f)$ is an affine variety.

Pf Go from special cases to general case:

- $f = x_0, X = \mathbb{P}^n \rightsquigarrow X \setminus V(x_0) = \mathbb{A}^n$ is affine var.
- f any linear polynomial, $X = \mathbb{P}^n$:

$f(x) = a_0 x_0 + a_1 x_1 + \dots + a_n x_n \rightsquigarrow$ Choose matrix

extend first row to basis

$$A = \begin{pmatrix} a_0 & a_1 & \cdots & a_n \\ & & ? & \end{pmatrix} \in GL(n+1, K)$$

[Exa 7.5(a)] $\rightsquigarrow$ $A : \mathbb{P}^n \xrightarrow{\sim} \mathbb{P}^n$ projective automorphism

$$X \setminus V(f) = \{(x_0:\dots:x_n) : f(x) \neq 0\} \overset{A^{-1}}{\longleftarrow} \{(y_0:\dots:y_n) : y_0 \neq 0\} \cong \mathbb{A}^n$$

still affine.

• f homogeneous of degree d, $X \subseteq \mathbb{P}^n$ proj. variety. Let $g \in K[y_0, \dots, y_N]$ be linear polynomial in Veronese coordinates corresp. to f

$$\rightsquigarrow \quad \begin{tikzcd} & \mathbb{P}^n \arrow[r, "F", "\sim"'] & Y \arrow[r, phantom, "\overset{\text{closed}}{\subseteq}"] & \mathbb{P}^N \\ X \setminus V(f) \arrow[r, phantom, "\underset{\text{closed}}{\subseteq}"] & \mathbb{P}^n \setminus V(f) \arrow[u, phantom, "\cup"] \arrow[r, "\sim"] & Y \setminus V(g) \arrow[u, phantom, "\cup"] \arrow[r, phantom, "\underset{\text{closed}}{\subseteq}"] & \underbrace{\mathbb{P}^N \setminus V(g)}_{\text{affine by proof above}} \end{tikzcd} \quad \rightsquigarrow F^{-1}(V(g)) = V(f)$$

$\Rightarrow Y \setminus V(g)$ affine (closed in $\mathbb{P}^N \setminus V(g)$)
$\Rightarrow X \setminus V(f)$ affine (closed in $\mathbb{P}^n \setminus V(f)$). $\square$

Exa